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Algebraic characterization of X -states in quantum information

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Online at stacks.iop.org/JPhysA/42/412002**Abstract**

A class of two-qubit states called X -states are increasingly being used to discuss entanglement and other quantum correlations in the field of quantum information. Maximally entangled Bell states and ‘Werner’ states are subsets of them. Apart from being so named because their density matrix looks like the letter X , there is not as yet any characterization of them. The $su(2) \times su(2) \times u(1)$ subalgebra of the full $su(4)$ algebra of two qubits is pointed out as the underlying invariance of this class of states. X -states are a seven-parameter family associated with this subalgebra of seven operators. This recognition provides a route to preparing such states and also a convenient algebraic procedure for analytically calculating their properties. At the same time, it points to other groups of seven-parameter states that, while not at first sight appearing similar, are also invariant under the same subalgebra. And it opens the way to analyzing invariant states of other subalgebras in bipartite systems.

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1. Introduction

Increasingly in the field of quantum information, aspects of entanglement [1], and of other quantum correlations such as, for instance, ‘quantum discord’ [2–4], between two qubits have been described for a class of pure and mixed states that have come to be called ‘ X -states’ [5–7]. Although their use goes back further [8–12], they were so named in [7] because of the visual appearance of the density matrix, that it looks like the letter in the alphabet:

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \quad (1)$$

Non-zero entries occur only along the diagonal and anti-diagonal. Many calculations of entanglement and other properties [8–16], and their evolution under unitary or dissipative

processes [17–25], can be carried out analytically for such states which make them appealing objects for study. Many other states of interest, such as the maximally entangled Bell states [1] and ‘Werner’ states [26], are a sub-class of X -states, lending further importance to their study. A more general class, depending on three rather than the one-parameter family of Werner states, for which quantum discord has recently been calculated [14, 27] are also a subset of X -states.

Yet, no firmer definition has been given of what makes a pure or mixed system an X -state. This paper provides such a definition in terms of their invariance properties, that a particular symmetry group or algebra underlies them. Such an identification of an underlying symmetry helps to explain the analytical results while at the same time providing a well-defined procedure for their preparation. Recognizing the symmetry also makes computations involving such states, such as unitary operations on them or evaluating concurrence or other measures of entanglement, straightforward and easily tractable. And, finally, the symmetry also opens the way for constructing other density matrices which may not visually appear as X , but are nevertheless similar, states of a different rendering of the same algebraic symmetry. Since they differ in entanglement and separability considerations, they may prove useful for study.

2. The subalgebra of X -states

Positivity and other standard requirements of any density matrix make the X -states shown in equation (1) a seven-parameter family. The diagonal elements of the density matrix are real so that, along with the trace being fixed at 1, three real parameters describe those diagonal entries. Hermiticity to guarantee real eigenvalues reduces the off-diagonal entries to two complex (say ρ_{14} and ρ_{23} , with ρ_{41} and ρ_{32} their respective complex conjugates) or four real parameters for the total of seven real parameters.

The full two-qubit system has the symmetry of the $SU(4)$ group and its algebra $su(4)$. Fifteen operators, most conveniently rendered as 15 linearly independent 4×4 matrices or as Pauli spinors/matrices of the two spins, together with the unit matrix, provide a complete description of the general system. There are, however, several subalgebras of $su(4)$. A series of recent papers have provided a geometrical description of their states and operators [28–33]. In particular, one subalgebra, $su(2) \times su(2) \times u(1)$, of seven operators or matrices occurs in many physical systems in quantum optics and quantum information [28, 29]. This paper presents them as the invariance set of the X -states.

Inspection of the explicit 4×4 matrices in a standard basis for two spins, ($|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$), is instructive [28, 29, 34, 35] and points immediately to sets of seven of them with the same structure of eight zeros in the same positions as in equation (1). That is, these are the operators that do not mix the 1–4 and 2–3 subspaces of the density matrix. Combined with the observation that such a set of seven matrices closes under multiplication, it is immediate that they will carry X -states into each other, that they preserve the X structure. For this purpose, both the Lie algebra aspect that the seven operators close under commutation and their Clifford algebraic structure that they close under multiplication are important. Indeed, explicit rendering of the 15 operators in terms of two Pauli spinors called $\vec{\sigma}$ and $\vec{\tau}$, together with the familiar multiplication rule $\sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k$, $i, j, k = 1-3$, where ϵ_{ijk} is the completely antisymmetric symbol and repeated indices are summed, is very useful for operations with them.

There are many such sets of seven operators/matrices constituting the $su(2) \times su(2) \times u(1)$ subalgebra [28, 29, 33]. In each of them, one operator, the $u(1)$ element, commutes with all six of the others which themselves can be further subdivided as shown in [28] into two sets

of ‘pseudospins’, two sets of three which obey commutation relations of angular momentum within each set while all three of one set commute with all three of the other. Any one of the 15 operators can serve as the commuting element because, as shown in a table in [29], each row has six zeros so that each identifies such a $su(2) \times su(2) \times u(1)$ set. There are, therefore, 15 such non-equivalent subalgebras.

We will designate such a set by $\{X_i\}$, $i = 1, 2, \dots, 7$, with X_1 the commuting element. One such is $(X_1 = \sigma_z \tau_z, X_2 = \sigma_y \tau_x, X_3 = \tau_z, X_4 = -\sigma_y \tau_y, X_5 = \sigma_x \tau_y, X_6 = \sigma_z, X_7 = \sigma_x \tau_x)$. This is the same set that occurs in the CNOT quantum logic gate constructed out of two Josephson junctions and was extensively studied in that context [28]. It was also pointed out that it occurs in nuclear magnetic resonance when each spin is in an external magnetic field in the z -direction while being coupled to each other through scalar coupling $\vec{\sigma} \cdot \vec{\tau}$ and ‘cross-coherences’ $\sigma_x \tau_y$ and $\sigma_y \tau_x$. But a different choice for the commuting element X_1 gives another such subalgebra, and we will return to this in section 4. Each X_i is traceless, Hermitian, and unitary, and its square is unity so that the eigenvalues are $(\pm 1, \pm 1)$.

With any such set $\{X_i\}$, the density matrix that remains invariant under their operations can be rendered as a linear superposition of them,

$$\rho = (I + \sum_i g_i X_i)/4, \quad (2)$$

in analogy to that for a single spin, $(I + \sum_i g_i \sigma_i)/2$. The seven real coefficients g_i in equation (2) parametrize X -states and are equivalent to the seven parameters in the density matrix in equation (1):

$$\begin{aligned} g_1 &= (\rho_{11} + \rho_{44}) - (\rho_{22} + \rho_{33}), \\ g_2 &= 2i(\rho_{14} - \rho_{41} + \rho_{32} - \rho_{23}), \\ g_3 &= (\rho_{11} - \rho_{44}) - (\rho_{22} - \rho_{33}), \\ g_4 &= 2(\rho_{14} + \rho_{41} - \rho_{32} - \rho_{23}), \\ g_5 &= 2i(\rho_{14} - \rho_{41} - \rho_{32} + \rho_{23}), \\ g_6 &= (\rho_{11} - \rho_{44}) + (\rho_{22} - \rho_{33}), \\ g_7 &= 2(\rho_{14} + \rho_{41} + \rho_{32} + \rho_{23}). \end{aligned} \quad (3)$$

The algebra of the seven $\{X_i\}$ is most conveniently captured by figure 1 as has recently been pointed out [33]. This figure occurs in projective geometry as the ‘Fano Plane’ [36] and also is used to represent the multiplication table for octonions [37–40]. Arranging the seven operators at the vertices, mid-points of sides and in-center of an equilateral triangle, the seven lines shown (including the inscribed circle) each pass through three points, providing the multiplication rule for those $\{X_i\}$. The notation of arrows is also borrowed from octonions except that unlike them which have all seven lines arrowed, the three internal verticals are not in figure 1. On those lines, all three operators mutually commute, so that the product of two gives the third regardless of order. On the four arrowed lines, the operators mutually anticommute so that the product of two gives $(\pm i)$ times the third, with plus (minus) signs along (against) the sense of the arrow. For this purpose, each line is regarded as a closed loop with a continuously circulating arrow. The central element commutes with all six of the others. For each of those, there is one ‘conjugate’ element with which it commutes and four with which it anticommutes. All of this can be read off by merely glancing at figure 1 and will provide simple rules for their manipulation in the next section.

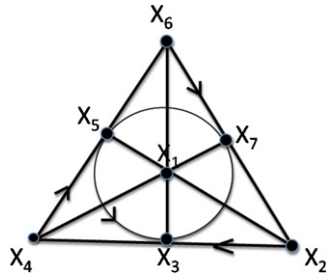


Figure 1. The multiplication diagram for the seven operators that underlie X -states. Resembling the Fano Plane and the multiplication diagram for octonions, each operator stands on three lines, and each line, including the inscribed circle, has on it three operators. On the interior verticals, the product of any two operators gives the third, these objects commuting. On the remaining four lines, the operators anticommute, and the product of any two gives the third with a multiplicative $\pm i$, the plus (minus) depending on the direction of (along/against) the arrow [33].

3. Manipulating density matrices with the elements of the subalgebra

The Clifford algebra structure and its diagrammatic rendering in figure 1 make operations on the density matrix of X -states very simple. Thus, because X_1 commutes with all the operators in equation (2), $X_1\rho X_1^\dagger$ leaves that ρ unchanged. For any of the other six cases, $X_i\rho X_i^\dagger$, three coefficients in equation (2) remain unchanged (those belonging to that i , its conjugate, and 1, that is, sharing an unarrowsed line in figure 1) while the other four are switched to their negative. As an example of another common operation which occurs, for instance, in the evaluation of ‘concurrence’ [41, 42] or ‘quantum discord’ [2], $\tilde{\rho} = \sigma_y\tau_y\rho^*\sigma_y\tau_y$ can also be written down from equation (2) without any calculation. This involves $X_4 (= -\sigma_y\tau_y)$. In figure 1, this element is connected to $i = 2, 3, 5, 6$ by arrows so that those coefficients g_i have their signs changed by this operation. But the additional complex conjugation involved changes signs again for $i = 2, 5$ whose X_i in equation (2) are pure imaginary for this set: $X_2 = \sigma_y\tau_x$, $X_5 = \sigma_x\tau_y$. As a result, $\tilde{\rho}$ differs from ρ in equation (2) by just flipping the sign of $i = 3, 6$. In like manner, all such operations become almost automated.

Next, in evaluating the concurrence, one needs the eigenvalues of $\rho\tilde{\rho}$. With the products of $\{X_i\}$ staying within the algebra and again easily written down from figure 1, $\rho\tilde{\rho}$ is again a linear combination of the form of equation (2). The square roots of the four eigenvalues of this matrix turn out to be

$$\begin{aligned} & \frac{1}{4}[\sqrt{(1 + g_1)^2 - (g_3 + g_6)^2} \pm \sqrt{((g_2 + g_5)^2 + (g_4 + g_7)^2)}] \\ & \frac{1}{4}[\sqrt{(1 - g_1)^2 - (g_3 - g_6)^2} \pm \sqrt{((g_2 - g_5)^2 + (g_4 - g_7)^2)}]. \end{aligned} \tag{4}$$

Note the appearance of the combinations of conjugate pairs 2–5, 3–6 and 4–7 of figure 1 (and, of course, 1 with the unit element). Arranging the quantities in equation (4) in decreasing order and subtracting the sum of the last three from the highest provides the concurrence for any X -state.

A prescription for creating X -states from a general density matrix can also be easily provided. Each of the 15 operators of $su(4)$ commutes with six and anticommutes with eight of the others [28, 29]. Thus, X_1 which commutes with the six in the $su(2) \times su(2) \times u(1)$ set necessarily anticommutes with all the eight left out of this set. As a result, in a general density matrix ρ of two qubits with all elements nonzero, action by this element, $X_1\rho X_1^\dagger$, changes the signs of the entries in those eight positions which have zeros in equation (1), while leaving other coefficients unchanged. As a result, starting from any general ρ , adding to it

$X_1 \rho X_1^\dagger$ will generate an X -state. Thus, X_1 is the ‘generator’ of its associated X -states and also provides an operational way of making the state, starting from a general density matrix. The action of $X_1 (= \sigma_z \tau_z)$ is, of course, the simultaneous unitary rotation through π of both spins achieved by a suitable magnetic field. Similar physical operations are easily described for other realizations of a two-level system or qubit in place of spins. In this manner, a convenient procedure for experimentally realizing X -states can be given.

In studying the dynamics of two qubits, such as entanglement evolution in the presence of dissipative processes, Kraus or other operators are involved. With these also expressible in terms of the set $\{X_i\}$, the X -structure of the states is preserved, as has been exploited in these studies of the finite-time end of entanglement and methods to overcome it [17–25].

4. Other classes of ‘ X -states’

The identification above of the ‘standard’ (as commonly used) X -states with the element $X_1 = \sigma_z \tau_z$ immediately points to several other groups of states having the same characteristic of an invariant algebra of $su(2) \times su(2) \times u(1)$ but with different choices for X_1 . As already observed, any of the 15 operators of $su(4)$ can be chosen for X_1 . One, for instance, is to choose for this element σ_z . The other six members of this $su(2) \times su(2) \times u(1)$ subalgebra are $(\vec{\tau}, \sigma_z \vec{\tau})$. All these matrices and the resulting density matrix that is invariant in form under their operations are now block diagonal 4×4 matrices, with all eight elements in the two off-diagonal blocks zero:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & \rho_{34} \\ 0 & 0 & \rho_{43} & \rho_{44} \end{pmatrix}. \tag{5}$$

While not looking like the X in equation (1), they now stand for a decoupling of the 1–2 and 3–4 subspaces, different from that in equation (1). In terms of the basis states, they can now be very different in separability and entanglement properties, grouping $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle)$ and $(|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle)$ together instead of $(|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle)$ and $(|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle)$ as in maximally entangled Bell states. Nevertheless, algebraically they are also invariant sets of an $su(2) \times su(2) \times u(1)$ algebra.

Even more interestingly, an X -state need not have any zeros in its density matrix! Thus, the choice $(X_1 = \sigma_x \tau_x, X_2 = \sigma_z \tau_y, X_3 = \tau_x, X_4 = -\sigma_z \tau_z, X_5 = \sigma_y \tau_z, X_6 = \sigma_x, X_7 = \sigma_y \tau_y)$ built on commuting element $X_1 = \sigma_x \tau_x$ is equally valid as an X -state with the same invariance algebra, although its density matrix has no zero entries:

$$\rho = \frac{1}{4}I + \frac{1}{4} \begin{pmatrix} -g_4 & g_3 - ig_2 & g_6 - ig_5 & g_1 - g_7 \\ g_3 + ig_2 & g_4 & g_1 + g_7 & g_6 + ig_5 \\ g_6 + ig_5 & g_1 + g_7 & g_4 & g_3 + ig_2 \\ g_1 - g_7 & g_6 - ig_5 & g_3 - ig_2 & -g_4 \end{pmatrix}. \tag{6}$$

Its preparation, by adding to ρ a transformation under $X_1 (= \sigma_x \tau_x)$, and its manipulation or calculation of concurrence and other properties, all proceed as in section 3, the density matrix staying within this $su(2) \times su(2) \times u(1)$ subalgebra. Indeed, as can be seen by comparing the two sets of $\{X_i\}$, it differs from the ‘standard’ set merely by a cyclic permutation of the indices (x, y, z) which means two $\pi/2$ rotations of axes, first clockwise with respect to y and then counter-clockwise with respect to an intermediate z axis. Alternatively, this can be

represented as the application of π pulses to the spins. Clearly, the physics is unchanged with mere change of bases or permutation of the labels of the coordinate axes.

Entanglement, discord [27] and other investigations of these other sets will be of interest. A natural extension for further investigation is to analyze similarly sets of invariant states of other subalgebras of $su(4)$ such as $su(3)$ and $so(5)$ [30–32]. The recognition of invariant sets opens a new window into such studies, focusing on what is essential and what are simply changes in bases and representations.

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